A new method of surface modeling and its application to DEM construction

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Abstract

A new method of surface modelling based on the fundamental theorem of surfaces (SMTS) is presented. Eight different test surfaces are employed to comparatively analyze the simulation errors of SMTS and the classical methods of surface modeling in GIS, including TLI (triangulated irregular network with linear interpolation), SPLINE, IDW (inverse distance weighted) and KRINGING. Numerical tests show that SMTS is much more accurate than the classical methods. SMTS theoretically gives a solution to the error problem that has long troubled DEM construction. As a real-world example, SMTS is used to construct a DEM of the Da-Fo-Si coal mine in Shaan-Xi Province, China. Its root mean square error (RMSE) is compared with those of DEMs constructed by the four classical methods. The results show that although SMTS also has a higher accuracy in the real-world example, the improvement of accuracy is less distinct than that expected from the numerical tests. The accuracy loss seems to be caused by location differences between sampling points and the central points of lattices of the simulated surfaces. Two alternative ways are proposed to solve this problem.

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1. Introduction

A digital elevation model (DEM) is a representation of terrain elevation as a function of geographic location. Discrete DEMs sampled on a regularly spaced lattice, known as gridded DEMs, can be easily stored and manipulated in digital form. They provide basic information required to characterize the topographic attributes of terrain (Hutchinson and Gallant, 2000; Pike, 2000). Any DEM, even a very high quality one, is an approximation to a real-world continuous surface (Carter, 1988). The sources of DEM errors include the quality of source data, data-capturing equipments, the transformation method of control points, the mathematical model for constructing the surface, and grid resolution and orientation (Zhou and Liu, 2002; Zhu et al., 2005). Such DEM errors can be propagated through simulation processes and affect the quality of final products (Huang and Lees, 2005; Oksanen and Sarjakoski, 2005). Although there are

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many types and sources of error and uncertainty in geographical data and their processing, the problem is not simply technical and may arise from an evident inability of GIS (Unwin, 1995).

A large number of studies have focused on the solutions of DEM errors. For instance, Goodchild (1982) introduced fractal Brownian process as a terrain simulation model for improving DEM accuracy. Walsh et al. (1987) found that the total error may be minimized by recognizing both inherent errors within input products and operational errors created by combinations of input data. Hutchinson and Dowling (1991) introduced a drainage enforcement algorithm that tried to remove spurious pits in order to yield a DEM that would reflect the natural drainage structure. Unwin (1995) suggested that it would be obviously DEM that would reflect the natural drainage structure. that tried to remove spurious pits in order to yield a (1991) introduced a drainage enforcement algorithm to reduce DEM errors. Podobnikar (2005) proposed to propagate through GIS operations were to be available, useful if general tools for examining how errors solutions of DEM errors. For instance, Goodchild (2005) investigated high-order interpolation algorithms errors caused by the Gibbs phenomenon. Shi et al. (1987) found that the total error may be minimized by recognizing both inherent errors within input products and operational errors created by combinations of input data. Florinsky (2002) argued that a clear distinction needs to be made between the lattice and pixel models when current GIS control system (Lo and Yeung, 2002). Hutchinson and Dowling (2000) argued that a clear distinction needs to be made inspection as the principle items of its DEM quality logical and physical formats of data files, and visual adopted statistical accuracy tests, data editing, verification of United States Geological Survey (1997) adopted extent of the grid, while the value stored in a lattice relates solely to the central point of the grid. The United States Geological Survey (1997) adopted statistical accuracy tests, data editing, verification of logical and physical formats of data files, and visual inspection as the principle items of its DEM quality control system (Lo and Yeung, 2002). Florinsky (2002) suggested four alternative methods to reduce DEM errors caused by the Gibbs phenomenon. Shi et al. (2005) investigated high-order interpolation algorithms to reduce DEM errors. Podobnikar (2005) proposed to produce high quality DEMs by using all available data sources, even lower quality datasets and datasets without a height attribute. Chaplot et al. (2006) evaluated the performance of various classic techniques for DEM generation. However, more studies are needed to attack the error problem at the root.

To find a solution for the DEM error problem, a new method of surface modeling based on the fundamental theorem of surfaces (SMTS) is developed. The fundamental theorem of surfaces assures that a surface is uniquely defined by the first and second fundamental coefficients (Henderson, 1998). If a surface is a graph of a function \( z = f(x, y) \) or \( r = (x, y, f(x, y)) \), the first fundamental coefficients, \( E, F \) and \( G \), can be formulated as,

\[
E = 1 + f_x^2,
\]

\[
F = f_x f_y,
\]

\[
G = 1 + f_y^2.
\]

The second fundamental coefficients, \( L, M \) and \( N \), can be formulated as,

\[
L = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}},
\]

\[
M = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}},
\]

\[
N = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}.
\]

The first and second fundamental coefficients satisfy the Gauss–Codazzi equations (Somasundaram, 2005). The Gauss equation can be formulated as,

\[
f_{xx} = \Gamma_{11}^1 f_x + \Gamma_{11}^2 f_y + L(EG-F^2)^{-\frac{3}{2}}
\]

\[
f_{xy} = \Gamma_{12}^1 f_x + \Gamma_{12}^2 f_y + M(EG-F^2)^{-\frac{3}{2}}
\]

\[
f_{yy} = \Gamma_{22}^1 f_x + \Gamma_{22}^2 f_y + N(EG-F^2)^{-\frac{3}{2}}.
\]

The Codazzi equation can be formulated as,

\[
\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x} = L \Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N \Gamma_{11}^2
\]
\[
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = L \Gamma_{12}^1 + M (\Gamma_{22}^2 - \Gamma_{12}^1) - N \Gamma_{12}^2
\]

(11)

where

\[
\begin{align*}
\Gamma_{11}^1 &= \frac{1}{2} \left( GE_x - 2FF_x + FE_y \right) \left( EG - F^2 \right)^{-1} \\
\Gamma_{12}^1 &= \frac{1}{2} \left( GE_y - FG_x \right) \left( EG - F^2 \right)^{-1} \\
\Gamma_{22}^1 &= \frac{1}{2} \left( 2GF_y - GG_x - FG_y \right) \left( EG - F^2 \right)^{-1} \\
\Gamma_{11}^2 &= \frac{1}{2} \left( 2EF_y - EE_x - FE_y \right) \left( EG - F^2 \right)^{-1} \\
\Gamma_{22}^2 &= \frac{1}{2} \left( EG_y - 2FF_y + FG_y \right) \left( EG - F^2 \right)^{-1}.
\end{align*}
\]

Our previous studies (Yue et al., 2004; Yue and Du, 2005, 2006) show that seven different surface models could be developed using different combinations of Eqs. (7)–(11). Errors and computing times of these surface models differ from each other, among which the one with the least error and less computing time (i.e., SMTS) could be formulated as (Yue and Du, 2006),

\[
f_{xx} = \Gamma_{11}^1 f_x + \Gamma_{11}^2 f_y + L \left( EG - F^2 \right)^{-\frac{1}{2}}
\]

(18)

\[
f_{yy} = \Gamma_{22}^1 f_x + \Gamma_{22}^2 f_y + N \left( EG - F^2 \right)^{-\frac{1}{2}}.
\]

(19)

If 1) \(\{(x_i, y_j)\}\) is a network created by orthogonal division of a computational domain \(\Omega\), 2) \(h_{ix} = x_{i+1} - x_{i-1}\) and \(h_{iy} = y_{j+1} - y_{j-1}\) are respectively grid spacing at a grid cell \((x_i, y_j)\) in directions \(x\) and \(y\), 3) \(\tilde{f}_{ij}\) is interpolation approximate value at the same grid cell in terms of sampling data \((x_{i}, y_{j}, f_{ij}) \in \Phi\), and 4) \(f_{ij} = \tilde{f}_{ij}\), then the finite difference of the first fundamental coefficients can be formulated as,

\[
E_{ij} = 1 + \left( \frac{\tilde{f}_{i+1,j} - \tilde{f}_{i-1,j}}{2h_{ix}} \right)^2
\]

(20)

\[
F_{ij} = \left( \frac{\tilde{f}_{i+1,j} - \tilde{f}_{i-1,j}}{2h_{ix}} \right) \left( \frac{\tilde{f}_{ij+1} - \tilde{f}_{ij-1}}{2h_{iy}} \right)
\]

(21)

\[G_{ij} = 1 + \left( \frac{\tilde{f}_{i+1,j} - \tilde{f}_{i-1,j}}{2h_{iy}} \right)^2.
\]

(22)

The finite difference of the second fundamental coefficients can be formulated as,

\[
L_{ij} = \frac{\tilde{f}_{i+1,j} - 2\tilde{f}_{i,j} + \tilde{f}_{i-1,j}}{h_{iy}^2}
\]

(23)

\[
M_{ij} = \frac{\tilde{f}_{i,j+1} - 2\tilde{f}_{i,j} + \tilde{f}_{i,j-1}}{h_{iy}^2} + \left( \frac{\tilde{f}_{i+1,j} - \tilde{f}_{i-1,j}}{2h_{iy}} \right)^2
\]

(24)

\[
N_{ij} = \frac{\tilde{f}_{ij+1} - 2\tilde{f}_{ij} + \tilde{f}_{ij-1}}{h_{ix}^2}
\]

(25)

The finite difference of \(\Gamma_k^1\) \((k=11, 12, 22; l=1, 2)\) can be formulated as,

\[
\begin{align*}
(\Gamma_{12}^1)_{ij} &= \frac{G_{ij}(E_{ij+1} - E_{ij-1}) h_{iy} - F_{ij}(G_{ij+1} - G_{ij-1}) h_{ix}}{4(E_{ij} G_{ij} - F_{ij}^2) h_{ix} h_{iy}} \\
(\Gamma_{12}^2)_{ij} &= \frac{E_{ij}(G_{ij+1} - G_{ij-1}) h_{ix} - F_{ij}(E_{ij+1} - E_{ij-1}) h_{ix}}{4(E_{ij} G_{ij} - F_{ij}^2) h_{ix} h_{iy}} \\
(\Gamma_{11}^1)_{ij} &= \frac{G_{ij}(E_{ij+1} - E_{ij-1}) h_{ix} - 2F_{ij}(F_{ij+1} - F_{ij-1}) h_{iy} + F_{ij}(E_{ij+1} - E_{ij-1}) h_{iy}}{4(E_{ij} G_{ij} - F_{ij}^2) h_{ix} h_{iy}} \\
(\Gamma_{12}^1)_{ij} &= \frac{2G_{ij}(F_{ij+1} - F_{ij-1}) h_{ix} - G_{ij}(G_{ij+1} - G_{ij-1}) h_{ix} - F_{ij}(G_{ij+1} - G_{ij-1}) h_{iy}}{4(E_{ij} G_{ij} - F_{ij}^2) h_{ix} h_{iy}} \\
(\Gamma_{11}^2)_{ij} &= \frac{2E_{ij}(F_{ij+1} - F_{ij-1}) h_{ix} - E_{ij}(E_{ij+1} - E_{ij-1}) h_{ix} - F_{ij}(E_{ij+1} - E_{ij-1}) h_{iy}}{4(E_{ij} G_{ij} - F_{ij}^2) h_{ix} h_{iy}}
\end{align*}
\]

(26) (27) (28) (29) (30) (31)
The finite difference of the SMPS can be formulated as,

\[
\begin{align*}
N_{ij} &= \frac{f^n_{ij+1} - f^n_{ij}}{h_{x_{i+1}} - h_{x_i}} + \frac{1}{2} \left( f^n_{ij+1} - f^n_{ij} \right) = \frac{f^n_{ij+1} - f^n_{ij}}{h_{x_{i+1}} - h_{x_i}} \\
+ \frac{1}{2} \left( f^n_{ij+1} - f^n_{ij} \right)
\end{align*}
\]

Based on the relevant method of numerical mathematics (Quarteroni et al., 2000), the iterative formulation of SMPS could be expressed as,

\[
\begin{align*}
N^n_{ij} &= \frac{f^n_{ij+1} - f^n_{ij}}{h_{x_{i+1}} - h_{x_i}} + \frac{1}{2} \left( f^n_{ij+1} - f^n_{ij} \right)
\end{align*}
\]

\[
\begin{align*}
N^n_{ij} &= \frac{f^n_{ij+1} - f^n_{ij}}{h_{x_{i+1}} - h_{x_i}} + \frac{1}{2} \left( f^n_{ij+1} - f^n_{ij} \right)
\end{align*}
\]

3. Numerical tests

3.1. SMPS simulation process

The SMPS uses existing data, as points or contours, to globally fit a surface through several iterative simulation steps. This surface is then used to interpolate a height value at an unknown point. The unknown points are located on a regularly spaced lattice and the end result is a gridded digital terrain model. The iterative simulation steps are summarized as follows: 1) normalizing the computational domain \( \Omega \) and the normalized computational domain \( \Omega \subset [0,1] \times [0,1] \); 2) performing interpolation on the normalized computational domain in terms of sampling data \((x_i, y_j, f_{ij})\), from which we can get interpolated approximate values \( \{ \tilde{f}_{ij} \} \) at point \((x, y)\); 3) letting \( f^n_{ij} = \tilde{f}_{ij} \) and calculating the first fundamental coefficients \( E^n_{ij} \), \( F^n_{ij} \) and \( G^n_{ij} \) and the second fundamental coefficients \( L^n_{ij} \) and \( N^n_{ij} \) as well as the coefficients of the SMPS equations in terms of \( \{ f^n_{ij} \} \); 4) for \( n \geq 0 \), we can get \( \{ f^{n+1}_{ij} \} \) by solving the SMPS equations; and 5) the iterative process is repeated until simulation accuracy is satisfied.

3.2. Comparative analysis of simulation errors

In addition to the newly developed SMPS, there are many other methods of surface modelling, which are widely used in various GIS applications (Isaaks and Srivastava, 1989; Cressici, 1993), such as TLI, IDW, KRINGING and SPLINE. These four classical surface
Test surface SMTS SPLINE KRIGING TLI IDW

<table>
<thead>
<tr>
<th>Test surface</th>
<th>SMTS</th>
<th>SPLINE</th>
<th>KRIGING</th>
<th>TLI</th>
<th>IDW</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_1(x, y)</td>
<td>1.2463 × 10^{-3}</td>
<td>3.9862 × 10^{-3}</td>
<td>1.4295 × 10^{-1}</td>
<td>2.8509 × 10^{-2}</td>
<td>1.4030 × 10^{-1}</td>
</tr>
<tr>
<td>f_2(x, y)</td>
<td>2.3406 × 10^{-4}</td>
<td>1.9878 × 10^{-3}</td>
<td>4.7567 × 10^{-3}</td>
<td>9.0201 × 10^{-3}</td>
<td>4.6162 × 10^{-2}</td>
</tr>
<tr>
<td>f_3(x, y)</td>
<td>4.7998 × 10^{-5}</td>
<td>6.5524 × 10^{-4}</td>
<td>2.4732 × 10^{-3}</td>
<td>4.1618 × 10^{-3}</td>
<td>3.9796 × 10^{-3}</td>
</tr>
<tr>
<td>f_4(x, y)</td>
<td>5.8282 × 10^{-5}</td>
<td>4.3721 × 10^{-4}</td>
<td>1.2908 × 10^{-3}</td>
<td>2.4557 × 10^{-3}</td>
<td>1.3953 × 10^{-2}</td>
</tr>
<tr>
<td>f_5(x, y)</td>
<td>2.8501 × 10^{-5}</td>
<td>1.0962 × 10^{-4}</td>
<td>5.0507 × 10^{-4}</td>
<td>9.9612 × 10^{-4}</td>
<td>4.8667 × 10^{-3}</td>
</tr>
<tr>
<td>f_6(x, y)</td>
<td>1.8963 × 10^{-5}</td>
<td>1.0445 × 10^{-4}</td>
<td>2.5823 × 10^{-4}</td>
<td>4.9668 × 10^{-4}</td>
<td>3.9796 × 10^{-3}</td>
</tr>
<tr>
<td>f_7(x, y)</td>
<td>8.6858 × 10^{-6}</td>
<td>1.5837 × 10^{-4}</td>
<td>4.2607 × 10^{-4}</td>
<td>8.7978 × 10^{-4}</td>
<td>4.6944 × 10^{-3}</td>
</tr>
<tr>
<td>f_8(x, y)</td>
<td>5.7448 × 10^{-6}</td>
<td>6.5652 × 10^{-5}</td>
<td>2.3417 × 10^{-4}</td>
<td>4.0412 × 10^{-4}</td>
<td>3.7144 × 10^{-3}</td>
</tr>
</tbody>
</table>

Fig. 1. The eight test surfaces used for model evaluation. (a) f_1(x, y), (b) f_2(x, y), (c) f_3(x, y), (d) f_4(x, y), (e) f_5(x, y), (f) f_6(x, y), (g) f_7(x, y), (h) f_8(x, y).

models were employed to comparatively analyze SMTS errors. All the classical interpolation methods were performed using the module of 3D Analyst in ARCGIS 9.0, and the default parameters of the software were employed. For IDW, the power is 2, the search radius is variable, and the maximum number of the searched points is 12. For SPLINE, the REGULARIZED option is used, the weight is 0.1, and the number of the searched points is 12. For KRIGING, ordinary method is selected, the model of semivariogram is spherical, the search radius is variable, and the maximum number of the search points is 12.

Eight mathematic surfaces were taken as the test surfaces so that the ‘true’ value can be pre-determined to avoid uncertainty caused by uncontrollable data errors. We selected mathematical surfaces which represent different types of shapes (Fig. 1) with the following formulas,

\[ f_1(x, y) = \cos(10y) + \sin(10(x-y)) \]  
\[ f_2(x, y) = e^{-(5-10x)^2/2} + 0.75e^{-(5-10y)^2/2} + 0.75e^{-(5-10x)^2/2} + e^{-(5-10y)^2/2} \]  
\[ f_3(x, y) = \sin(2\pi y) \cdot \sin(\pi x) \]  
\[ f_4(x, y) = 0.75e^{-\left\{ (9x-2)^2 + (9y-2)^2 \right\}/4} + 0.75e^{-\left\{ (9x+1)^2 + (9y+1)^2 \right\}/10} + 0.5e^{-\left\{ (9x-7)^2 + (9y-3)^2 \right\}/4} - 0.2e^{-\left\{ (9x-4)^2 - (9y-7)^2 \right\}} \]  
\[ f_5(x, y) = \frac{1}{9} (\tanh(9y-9x) + 1) \]  
\[ f_6(x, y) = \frac{1.25 + \cos(5.4y)}{6 \left\{ 1 + (3x-1)^2 \right\}} \]  
\[ f_7(x, y) = \frac{1}{3} e^{-(81/16) \left\{ (x-0.5)^2 + (y-0.5)^2 \right\}} \]  
\[ f_8(x, y) = \frac{1}{3} e^{-(81/4) \left\{ (x-0.5)^2 + (y-0.5)^2 \right\}} \].

In the normalized computational domain, 14,641 (121 × 121) lattices were created by an orthogonal division of the computational domain. Therefore, in
simulation processes using SMTS and the classical interpolation methods, the grid spacing (or simulation step length) was set as $h_x = h_y = h = \frac{1}{121}$. Two sampling methods, uniform sampling and random sampling, were adopted to comparatively analyze errors of SMTS and the classical interpolation methods. For the uniform sampling, $5h$ was selected as the sampling interval to have 625 sampling points. For the random sampling, 1296 points were randomly sampled. The root mean-square error (RMSE) in this case is expressed as,

$$\text{RMSE} = \sqrt{\frac{1}{121 \cdot 121} \sum_{i=1}^{121} \sum_{j=1}^{121} (f_{ij} - Sf_{ij})^2}.$$  \hspace{1cm} (53)

where $Sf_{ij}$ is the numerical solution of $f(x,y)$ at point $(x_i, y_j)$; $f_{ij}$ is true value of $f$ at the lattice $(i,j)$.

Under uniform sampling (Tables 1 and 2), the accuracy of SMTS compared with the other methods is the lowest for the test surface $f_1(x,y)$ on average. However, the SMTS accuracy is still 3.85, 17.7, 35 and 171 times as high as the accuracy of SPLINE, KRIGING, TLI and IDW respectively. Test surface $f_8(x,y)$ gives SMTS the highest accuracy, which is 11.4, 40.8, 70.3 and 647 times as high as that of SPLINE, KRIGING, TLI and IDW respectively. On average, the accuracy of SMTS simulation is 9.15 times as high as that of SPLINE, 41.8 times of KRIGING, 53.9 times of TLI, and 276 times of IDW.

Under random sampling (Tables 3 and 4), SMTS simulation accuracies of the test surfaces $f_1(x,y)$, $f_2(x,y)$, $f_3(x,y)$, $f_4(x,y)$, $f_5(x,y)$, $f_6(x,y)$, $f_7(x,y)$ and $f_8(x,y)$ are respectively 47.1, 62.3, 77.7, 108, 56.8, 108, 206, and 198 times as much as the accuracies of the classical interpolation methods on average. The same method has different levels of power to simulate different test surfaces. For instance, when IDW is used to simulate the surface $f_8(x,y)$ under the random sampling, its RMSE is 621 times as much as that of SMTS, but RMSE of IDW is only 1.34 times as much as that of SMTS for $f_1(x,y)$.

The sampling methods greatly affect the accuracy of different interpolation methods simulating the particular test surface. For example, RMSE of IDW simulating the test surface $f_1(x,y)$ is only 1.34 times as much as that of SMTS under the random sampling, but 113 times under the uniform sampling. For the simulation of the test surface $f_3(x,y)$, the accuracy of SMTS under the random sampling is 5.67, 35.8, 36.1 and 233 times as high as that of SPLINE, KRIGING, TLI and IDW, respectively, but it is 15, 56.5, 95 and 90.9 times under the uniform sampling.

Many error assessments so far simply reported a single number to express the accuracy without regard for the location. If all errors balance out, non-site specific accuracy assessments give misleading results (Lunetta

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Ratio of RMSE of a classical method to that of SMTS under uniform sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test surface</td>
<td>SPLINE</td>
</tr>
<tr>
<td>$f_1(x,y)$</td>
<td>3.20</td>
</tr>
<tr>
<td>$f_2(x,y)$</td>
<td>8.49</td>
</tr>
<tr>
<td>$f_3(x,y)$</td>
<td>15</td>
</tr>
<tr>
<td>$f_4(x,y)$</td>
<td>7.5</td>
</tr>
<tr>
<td>$f_5(x,y)$</td>
<td>3.85</td>
</tr>
<tr>
<td>$f_6(x,y)$</td>
<td>5.51</td>
</tr>
<tr>
<td>$f_7(x,y)$</td>
<td>18.2</td>
</tr>
<tr>
<td>$f_8(x,y)$</td>
<td>11.4</td>
</tr>
<tr>
<td>On average</td>
<td>9.15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3</th>
<th>RMSE of simulation methods under random sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test surface</td>
<td>SMRTS</td>
</tr>
<tr>
<td>$f_1(x,y)$</td>
<td>$7.7636 \times 10^{-4}$</td>
</tr>
<tr>
<td>$f_2(x,y)$</td>
<td>$2.0898 \times 10^{-4}$</td>
</tr>
<tr>
<td>$f_3(x,y)$</td>
<td>$1.0532 \times 10^{-4}$</td>
</tr>
<tr>
<td>$f_4(x,y)$</td>
<td>$4.0324 \times 10^{-5}$</td>
</tr>
<tr>
<td>$f_5(x,y)$</td>
<td>$2.8466 \times 10^{-5}$</td>
</tr>
<tr>
<td>$f_6(x,y)$</td>
<td>$1.0014 \times 10^{-5}$</td>
</tr>
<tr>
<td>$f_7(x,y)$</td>
<td>$7.0024 \times 10^{-6}$</td>
</tr>
<tr>
<td>$f_8(x,y)$</td>
<td>$5.2814 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Ratio of RMSE of a classical method to that of SMTS under random sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test surface</td>
<td>SPLINE</td>
</tr>
<tr>
<td>$f_1(x,y)$</td>
<td>6.21</td>
</tr>
<tr>
<td>$f_2(x,y)$</td>
<td>7.91</td>
</tr>
<tr>
<td>$f_3(x,y)$</td>
<td>5.67</td>
</tr>
<tr>
<td>$f_4(x,y)$</td>
<td>9.56</td>
</tr>
<tr>
<td>$f_5(x,y)$</td>
<td>7.48</td>
</tr>
<tr>
<td>$f_6(x,y)$</td>
<td>7.99</td>
</tr>
<tr>
<td>$f_7(x,y)$</td>
<td>20.1</td>
</tr>
<tr>
<td>$f_8(x,y)$</td>
<td>10.1</td>
</tr>
<tr>
<td>On average</td>
<td>9.37</td>
</tr>
</tbody>
</table>
Fig. 2. Error distribution contours of simulations applied to $f_1(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) shows the contour chart of $f_1(x,y)$.

Fig. 3. Error distribution contours of simulations applied to $f_2(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) shows contour chart of $f_2(x,y)$.

Fig. 4. Error distribution contours of simulations applied to $f_3(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) shows contour chart of $f_3(x,y)$.

Fig. 5. Error distribution contours of simulations applied to $f_4(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) contour chart of $f_4(x,y)$.
Fig. 6. Error distribution contours of simulations applied to $f_5(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) contour chart of $f_5(x,y)$.

Fig. 7. Error distribution contours of simulations applied to $f_6(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) contour chart of $f_6(x,y)$.

Fig. 8. Error distribution contours of simulations applied to $f_7(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) contour chart of $f_7(x,y)$.

Fig. 9. Error distribution contours of simulations applied to $f_8(x,y)$. (a) SMTS, (b) SPLINE, (c) TLI, (d) KRIGING, (e) IDW, (f) contour chart of $f_8(x,y)$.
et al., 1991). Therefore, we introduce an error matrix for every method of surface modeling for each test surface, \( \{E_{ij}\} \). The error \( E_{ij} \) at the lattice point (or grid cell) \((i,j)\) is formulated as,

\[
E_{ij} = S_{fi} - f_{ij}
\]

where \( f_{ij} \) is the true value of \( f \) and \( S_{fi} \) is the simulated value of \( f \) at the lattice point \((i,j)\).

Figs. 2–9 indicate that the errors of SMTS are bigger along the boundaries of the resultant surfaces or in the areas near the peaks. If the test surfaces are symmetric,

Fig. 10. Location and topography of the Da-Fo-Si coal mine.

Fig. 11. Location of height sampling points of the Da-Fo-Si coal mine. a) Points for DEM construction, b) points for inspecting DEM accuracy.

Fig. 12. 3D shaded relief maps of the Da-Fo-Si coal mine, from DEMs constructed by a) SMTS, b) KRIGING, c) TLI, d) SPLINE, and e) IDW. View from west. Light from NW, with a dip angle of 45°.
the error distributions of SMTS, IDW, TLI and KRICING simulations are also symmetric, but the error distribution of SPLINE is asymmetric and tends to be more uniform.

4. A real-world example

Five sets of DEMs for the Da-Fo-Si coal mine in Shaan-Xi Province, China, were constructed using SMTS, KRIGING, TLI, SPLINE and IDW, and their errors were comparatively analyzed. The coal mine (Fig. 10) is located at N35°05′ and E108°00′ in northwestern Xian-Yang City, and its area is 1.4 km². A great topographical variety of the area is suitable for a study of DEM accuracy.

Twelve GPS (Global Positioning System) control points were allocated within the coal mine. The Leica TC403 total station was used to collect 3856 elevation points, of which 771 points (Fig. 11a) were randomly taken to construct DEMs, and the other 3085 points (Fig. 11b) were used to inspect the accuracy of the constructed DEMs.

DEMs of the coal mine with a spatial resolution of 1 m were constructed by the five methods. 3D shaded relief maps of the DEMs were used to represent the terrain relief of the coal mine (Fig. 12). The results show that the

Fig. 13. Comparisons between simulation results and observed data. a) SMTS, b) TLI, c) KRIGING, d) SPLINE, and e) IDW.
KRIGING simulation makes the DEM surface fragmental (Fig. 12b). The surface of TLI simulation within each triangle is a plane passing through the three vertices (Hugentobler et al., 2005), causing the so-called peak-truncation and pit-fill problem (Fig. 12c). The SPLINE simulation has a higher oscillation on the boundary of the DEM (Fig. 12d), and the IDW simulation produces many “Bull’s eyes” on the DEM surface (Fig. 12e). In contrast, the SMTS simulation provided a better result (Fig. 12a).

Fig. 13 shows the relationship between the height from the simulated DEM and the corresponding height of the observed point altitude. This analysis also indicates that SMTS simulation has the highest accuracy. TLI, KRIGING and IDW tend to make lower elevation higher and higher elevation lower. Many SPLINE simulation points are relatively far from the straight line of y=x. The results of accuracy assessment show that the mean absolute error of SMTS is 6.754 m, while the errors of KRIGING, TLI, SPLINE and IDW are 7.523, 7.697, 7.919 and 11.529 m, respectively.

5. Discussion and conclusions

Both the numerical tests and the real-world example demonstrate that the accuracy of SMTS is higher than that of the classical methods. In the numerical tests, various test surfaces, including symmetric, asymmetric, single-peak, multi-peak, gently changing and sharply changing ones, were simulated by SMTS and the classical methods. The comparative error analyses indicate that SMTS errors are much smaller than the ones of the classical methods.

In the real-world example, although the accuracy of SMTS was higher than the other four methods, the difference was less distinct than the case of the numerical tests. In other words, SMTS has accuracy loss in the practical application, which may have caused by location differences between the sampling points and the corresponding central points of lattices of the simulated surfaces. In the real-world example, most of the sampling points are not located at the centers of their corresponding lattices, and the nearest neighbor method was used to accommodate the location differences. This problem could be resolved by two alternative ways: 1) height sampling at the centers of the lattices through a careful design of investigation, and 2) establishing a continuous Taylor expansion of a DEM, \( z = f(x, y) \), on the basis of topographical characteristics.

SMTS has a huge computation cost because it must use an equation set for simulating each lattice of a surface. If a surface consists of 2.25 million grid cells and SMTS is operated on a typical personal computer available today, the simulation process needs about 10 h, and the computation time is approximately proportional to the third power of the total number of grid cells. A future research focus is to shorten computing time, by applying improved SMTS equations, the domain decomposition method (Toselli and Widlund, 2005), and parallel computing technology (Petersen and Arbenz, 2004).

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